Financial Mathematics – A spreadsheet based course

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Introduction

- These notes give background information useful in completing the exercises in this course.
- In general these notes do not repeat the outcomes of the exercises – that is left for the exercises themselves.

Introduction

- Consider these notes as "building blocks" to help you successfully complete the exercises and reveal their outcomes and insights.

The power function

- The power function is used when a quantity is multiplied by itself a number of times
- e.g. suppose 3.1 is multiplied by itself
- 3.1 * 3.1 is written more concisely as 3.1^2
- 3.1 * 3.1 is 3.1 to the power of 2
- 3^4 is 3 to the power of 4 = 3*3*3*3 = 81
- The power indicates the number of multiplications

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The power function

- $2^3 * 2^4 = 2*2*2 \quad * \quad 2*2*2*2 = 2^7$
- More generally, $a^n * a^m = a^{m+n}$ [P.1]
- $2^7 / 2^3 = 2*2*2*2*2*2*2 / (2*2*2)$
- $= 2*2*2*2 = 2^4$
- More generally, $a^n / a^m = a^{n-m}$ [P.2]

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The power function

- $2^2 = 4$, Consider the trend in dividing by 2
- $2^1 = 2$
- $2^0 = 1$
- $2^{-1} = ½ = 1/2^1$
- $2^{-2} = ¼ = 1/2^2$
- More generally, $a^{-m} = 1/a^m$ [P.3]
- and $a^0=1$ [P.4]

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The power function

- $(2^3)^4 = 2*2*2 \quad * \quad 2*2*2 \quad * \quad 2*2*2 \quad * \quad 2*2*2$
- $(2^3)^4 = 2^{12}$
- In general $(a^m)^n = a^{m*n}$ [P.5]

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The power function

- Powers can be fractional.
- What is $16^{0.25}$?
- Using [P.1] we can say this about $16^{0.25}$ ..
- $16^{0.25} * 16^{0.25} * 16^{0.25} * 16^{0.25} = 16^{4 * 0.25} =$
- $16^1 = 16$
- $16^{0.25}$ multiplied by itself 4 times is16

The power function

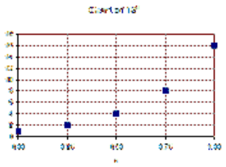
- Q. What, multiplied by itself four times is 16?
- A. 2 multiplied by itself four times is 16. i.e. 2*2*2*2 = 16. So $16^{0.25} = 2$
- Q. What is $16^{0.5}$?
- A. $16^{0.5} * 16^{0.5} = 16$.
- 4 * 4 = 16. So $16^{0.5} = 4$

The power function

- What is $16^{0.75}$?
- $16^{0.75} = (16)^{3/4} = (16)^{¼ * 3}$
- Using [P.5] we can say $(16)^{¼ * 3} = (16^{1/4})^3$
- $= 2^3$
- = 8

The power function

- Following is a chart showing the values we have calculated for 16^x as a function of x.

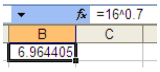


The power function

- While it is simple to calculate 16^x for certain x's (0, 0.25, 0.50, 0.75 and 1) for most other x's there is no simple way of calculating16^x by hand.
- For example, what is $16^{0.7}$?
- We can say that $16^{0.7}$ should be between $16^{0.5}$ (4) and $16^{0.75}$ (8) but we need a spreadsheet or calculator to obtain its precise value.

The power function

- The answer, to six decimal places, is 6.964405



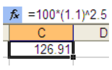
- As one would expect this answer is between $16^{0.5}$ (4) and $16^{0.75}$ (8)

The power function

- The power function is useful in compounding.
- e.g. $100 earns 10% interest compounding annually. How much is the compounded value at the end of two years?
- =100*(1+10%)*(1+10%)
- $=100*(1+10\%)^2 = 100*1.1^2 = 121$

The power function

- e.g. $100 earns 10% interest compounding annually. How much is the compounded value at the end of two and a half years?
- $=100*(1+10\%)^{2.5}$
- $=100*(1+10\%)^{2.5} = 100*1.1^{2.5} = 126.91$

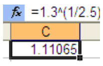


The power function

- Sometimes we need to “invert” the power function
- e.g. $a^3 = 7$, what is a?
- Raise both sides to the power of 1/3
- $(a^3)^{1/3} = 7^{1/3}$
- $a^{3/3} = a^1 = a = 7^{1/3} = 1.91$
- So a is 1.91

The power function

- What interest rate annually compounding for 2.5 years will make a $100 investment grow to $130?
- $100*(1+r)^{2.5} = 130$
- So $(1+r)^{2.5} = 1.3$
- So $(1+r) = 1.3^{1/2.5} = 1.11065$



- So r = 1.11065 – 1 = 11.065%

The log function

- The log function relates to powers of 10
- log (n) is the number of times 10 must be multiplied by itself to give n
- log(10,000) is 4 since $10,000 = 10*10*10*10 = 10^4$
- log(1,000) is 3 since $1,000 = 10*10*10 = 10^3$
- log(100) is 2 since $100 = 10*10 = 10^2$

The log function

- log(10) is 1 since $10^1 = 10$
- log(1) is 0 since $10^0 = 1$
- log(0.1) is -1 since $10^{-1} = 1/10^1 = 0.1$
- Saying the above in another way..
- $\log(10^2) = 2$; $\log(10^1) = 1$; $\log(10^3) = 3$;
- In general $\log(10^n) = n * \log(10) = n$
- More generally, $\log(a^n) = n * \log(a)$ [L.1]

The log function

- $\log(10^5 * 10^3) = \log(10^8) = 8$
- We can write 8 as 5 + 3
- $5 + 3 = \log(10^5) + \log(10^3)$
- So $\log(10^5 * 10^3) = \log(10^5) + \log(10^3)$
- In general, log(a * b) = log(a) + log(b) [L.2]

The log function

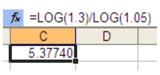
- $\log(10^8 / 10^3) = \log(10^5) = 5$
- We can write 5 as 8 - 3
- $8 - 3 = \log(10^8) - \log(10^3)$
- So $\log(10^8 / 10^3) = \log(10^8) - \log(10^3)$
- In general, log(a / b) = log(a) - log(b) [L.3]

The log function

- We can apply the log function to the following problem.
- $100 compounds annually at 5% p.a. How long will it take to compound to a value of $130?
- $100 * (1 + .05)^n = 130$
- So $(1 + .05)^n = 1.3$
- So $1.05^n = 1.3$

The log function

- Take the log of both sides
- $\log(1.05^n) = \log(1.3)$
- n * log(1.05) = log(1.3) [Using L.1]
- so n = log(1.3)/log(1.05) = 5.3774 years

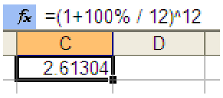


The exponential function

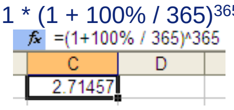
- Suppose $1 earns 100% simple interest for one year. How much will it be worth at the end of the year?
- 1 * (1 + 100%) = 1 * 2 = 2
- Suppose interest is compounded six-monthly. How much will the $1 be worth?
- $1 * (1 + 100\%/2)^2 = 1.5^2 = 2.25$

The exponential function

- Suppose compounding is monthly
- $1 * (1 + 100\% / 12)^{12} = 2.61304$



- What if compounding is daily?
- $1 * (1 + 100\% / 365)^{365} = 2.71457$



The exponential function

- In the limit, as compounding becomes continuous, $1 will compound (at 100% for one year) to 2.71828...
- This number occurs so often in mathematics that it is given the symbol e.

The exponential function

- So $1 compounding for one year (at 100%) will grow to e dollars.
- What if the $1 compounded for 2 years?
- In the first year it would grow by a factor of e and in the second year by another e.
- So at the end of two years it would be worth e^2.

The exponential function

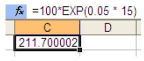
- What if the $1 compounded for t years?
- At the end of t years it would be worth e^t.
- Up to now we have been considering a compounding rate of 100%. What if the rate were 5%? What would the investment be worth at the end of t years?
- Instead of growing to e^t (e^{1*t}) it would grow to $e^{0.05*t}$.

The exponential function

- In general, $1 continuously compounding at r% for t years, would grow to e^{rt}.
- So, for example, 10 years compounding at 10% would grow to the same value as 20 years compounding 5%.
- In spreadsheets the exponential function exp is used to calculate powers of e.

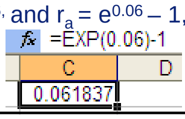
The exponential function

- Q. What would $100 compound to if the nominal annual continuously compounding rate is 5% and the term is 1.5 years?
- A. The following formula is used
- $100 * $e^{0.05 * 1.5}$
- The answer is 211.700002.



The exponential function

- Q. What annually compounding rate is equivalent to a continuously compounding rate of 6%?
- A. Let r_a be the annually compounding rate. Let t be the term of an investment.
- Then $(1+r_a)^t = e^{0.06*t}$, so $(1+r_a)^t = (e^{0.06})^t$
- So $1+r_a = e^{0.06,}$ and $r_a = e^{0.06} – 1$, and
- $r_a = 6.1837\%$

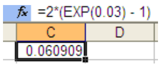


The exponential function

- Q. What semi-annually compounding rate is equivalent to a continuously compounding rate of 7%?
- A. Let r_s be the semi-annually compounding rate. Let t be the term of an investment.
- Then $(1+r_s/2)^{2*t} = e^{0.06*t}$
- So $\left[\left(1+\frac{r_s}{2}\right)^2\right]^t = \left[e^{0.06}\right]^t$

The exponential function

- We have $\left[\left(1+\frac{r_s}{2}\right)^2\right]^t = \left[e^{0.06}\right]^t$
- So $(1+r_s/2)^2 = e^{0.06}$
- So $(1+r_s/2) = e^{0.03}$
- So $r_s/2 = e^{0.03} - 1$
- So $r_s = 2 * (e^{0.03} - 1) = 6.0909\%$



The exponential function

- Following is an important property of the exponential function:

$$e^{\Delta} \approx 1 + \Delta$$

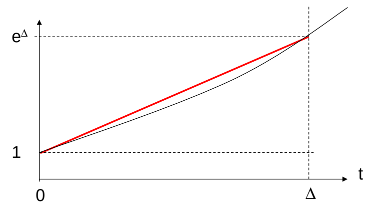
- In the formula above Δ represents a small number (e.g. 0.0001) and the "wavy" equals sign means "is approximately equal to".
- We'll show the above property is true by analysing a simple investment.

The exponential function

- Suppose we invest $1 at 100% continuously compounding.
- How much is our investment gaining in value (in dollars per year) at its inception?
- $1 is invested and the investment, over a small interval of time Δ, must be gaining value at $1 per year.
- We can show this graphically:

The exponential function

- The slope of the straight line is 1. But the slope is also equal to $(e^{\Delta}-1)/\Delta$.
- So $e^{\Delta} - 1 = \Delta$ and $e^{\Delta} = 1 + \Delta$



Sensitivity to interest rate

- What is the sensitivity of P to changes in interest rate? What will be the percentage change in P if the interest rate increases by Δ?
- The percentange change is:

$$\frac{P(r+\Delta)-P(r)}{P(r)}=\frac{e^{-(r+\Delta)t}-e^{-rt}}{e^{-rt}}$$

$$=\frac{e^{-rt}\left(e^{-\Delta t}-1\right)}{e^{-rt}}=-\Delta t$$

Sensitivity to interest rate

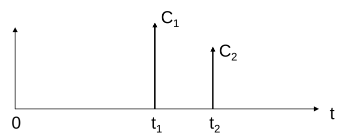
- So the percentage change is the change in interest rate multiplied the time at which the future cashflow occurs.
- If we define the interest rate sensitivity as being the percentage change in present value divided by the change in continuously compounding rate then the sensitivity is simply t – the time of the cashflow.

Sensitivity to interest rate

- Q. $100 is received 1.5 years in the future. The continuously compounded rate is 5%. What will be the percentage change in the present value of the $100 if the continuously compounded rate increases to 5.1%?
- A. % change = -time of cashflow * change in interest rate = 1.5 * (5.1%-5%)=-0.15%

Sensitivity to interest rate

- Suppose we have multiple cashflows as shown below. What is the sensitivity of their present value to changes in interest rate?



Sensitivity to interest rate

- If the continuously compounding rate is r then the present value V_1 of cashflow C1 is given by:

$$V_1 = C_1 e^{-rt_1}$$

- V_2 is given by:

$$V_2 = C_2 e^{-rt_2}$$

- And the total present value V is:

$$V = V_1 + V_2$$

Sensitivity to interest rate

- If r increases by Δ then the change in V_1 will be:

$$V_{1(r+\Delta)} - V_{1(r)} = -\Delta t_1 . V_{1(r)}$$

- Similarly, the change in V_2 will be:

$$V_{2(r+\Delta)} - V_{2(r)} = -\Delta t_2 . V_{2(r)}$$

- And the total change in V will be:

$$V_{(r+\Delta)} - V_{(r)} = -\Delta(t_1.V_1 + t_2.V_2)$$

Sensitivity to interest rate

- Dividing both sides by ΔV gives:

$$\frac{V_{(r+\Delta)} - V_{(r)}}{\Delta V_{(r)}} = -\frac{(t_1.V_1 + t_2.V_2)}{V_1 + V_2}$$

- The percentange change in present value divided by the change in interest rate (continuously compounded) equals the sum of the time-weighted present values divided by the sum of the present values.

Sensitivity to interest rate

- Q. $100 will be received in one year and $150 will be received in 1.5 years. If interest rates move from 6% to 6.1% what will be the percentage change in present value?
- A. The percentage change is given by:

$$\frac{V_{(r+\Delta)} - V_{(r)}}{V_{(r)}} = -\Delta \frac{(t_1.V_1 + t_2.V_2)}{V_1 + V_2}$$

Sensitivity to interest rate

- V_1 is $100*e^{-0.06*1} = 94.1765$
- V_2 is $150*e^{-0.06*1.5} = 137.0897$
- So the percentage change is:

$$-0.1\% * \left[\frac{1*94.1765 + 1.5*137.0897}{94.1765 + 137.0897}\right]$$
$$= -0.1\% * 1.2964$$
$$= -0.13\%$$

The ln (natural log) function

- Earlier we looked at the log function. The log of a number is the number of times 10 has to be multiplied by itself to give the number.
- For example, the log of 100 (i.e. log(100)) is 2 since 100 = 10*10.
- This log is a log to the base of 10. Logs can be to other bases.

The ln (natural log) function

- For example $\log_2$ (i.e. log to the base of 2) is the number of times 2 has to be multiplied by itself to give a number.
- So $\log_2$ (16) is 4 since 2 must be multiplied by itself 4 times to give 16.
- i.e. 2*2*2*2 = 16
- The logarithm to the base of e is especially useful and is written either as $\log_e$() or as ln(). This is called the natural log.

The ln (natural log) function

- The following table shows the natural logs of some powers of e.
- $\ln(e^3) = \ln(e*e*e) = 3$
- $\ln(e^2) = \ln(e*e) = 2$
- ln(e1) = ln(e) = 1
- $\ln(e^0) = \ln(1) = 0$
- $\ln(e^{-1}) = \ln(1/e^1) = -1$

The ln (natural log) function

- The ln function has similar properties to the $\log_{10}$ function. We saw earlier that ..
- $\log_{10}(a^n) = n * \log_{10}(a)$, similarly
- $\ln(a^n) = n * \ln(a)$
- We also saw that ..
- $\log_{10}(a * b) = \log_{10}(a) + \log_{10}(b)$, similarly
- $\ln(a * b) = \ln(a) + \ln(b)$

The ln (natural log) function

- Q. What continuously compounding rate is equivalent to an annually compounding rate of 5%?
- A. Let r_c be the continuous rate. Then

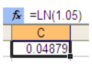
$$e^{r_c t} = (1 + 0.05)^t = 1.05^t$$

- So $e^{r_c} = 1.05$
- Take the ln of both sides

$$\ln\left(e^{r_c}\right) = \ln(1.05)$$

The ln (natural log) function

- But $\ln\left(e^{r_c}\right) = r_c \ln(e) = r_c$
- So $r_c = \ln(1.05) = 4.879\%$



Sensitivity to time

- Suppose we invest $1 at a continuously compounding rate of r%. We saw earlier that its V at time t is given by
- $V(t) = e^{rt}$
- At what rate is the investment gaining in value at time t?

Sensitivity to time

- The rate of value growth is given by:

$$\frac{V(t+\Delta)-V(t)}{\Delta}=\frac{e^{r(t+\Delta)}-e^{rt}}{\Delta}$$

$$=\frac{e^{rt}\left(e^{r\Delta}-1\right)}{\Delta}\approx\frac{e^{rt}(r\Delta)}{\Delta}=re^{rt}=rV(t)$$

- As one would expect – the rate at which the investment's value increases is its current value times the continuously compounding rate.

Sensitivity to time

- Q. $100 is invested and earns 8% interest compounding annually. At what rate is the investment growing in one year.
- A. Convert the annually compounding rate to a continuous rate r_c.
- r_c = ln(1.08) = 7.6961%
- The rate of growth is then r_c times the value in one year [108]. The answer is 7.6961% * 108 = $8.31 p.a.

Growth

- Suppose in successive years an investment grows by 5%, 4% and 3%.
- The mean (arithmetic) yearly growth rate will be [5% + 4% + 3%] / 3 = 4%.
- The CAGR (compound average growth rate) will be that rate, which compounded over three years, increases by a factor of 1.05 * 1.04 * 1.03.

Growth

- So in this case:
- $(1+CAGR)^3$ = 1.05 * 1.04 * 1.03
- 1 + CAGR = $(1.05 * 1.04 * 1.03)^{1/3}$
- CAGR = $(1.05 * 1.04 * 1.03)^{1/3}$-1
- CAGR = 3.9968%
- Note that this is less than the mean yearly growth of 4%.

Growth

- Suppose an investment of $100 earns a nominal 20% annual interest. Interest compounds annually. The investment is held for one year.
- At the end of one year the investment will have grown to $120.
- In this case both the nominal and effective annual rates are 20%

Growth

- Continuing the previous example – suppose interest compounds semi-annually. What is the annual effective rate?
- At the end of six months the investment will have compounded to:

$$100 * \left(1 + \frac{20\%}{2}\right)$$

Growth

- In another six months the investment will have compounded to:

$$100 * \left(1 + \frac{20\%}{2}\right) * \left(1 + \frac{20\%}{2}\right) =$$

$$100 * 1.1 * 1.1 = 100 * 1.21 = 121$$

- So a nominal annual rate of 20% compounding semi-annually is equivalent to an annual effective rate of 21%.

Growth

- The spreadsheet functions EFFECT and NOMINAL convert between effective and nominal rates.

Growth

- Function: EFFECT
- Purpose: To convert from nominal to effective rate
- Parameters:
  - =EFFECT(nominal_rate,npery)
  - nominal_rate is the nominal rate
  - npery is the number of periods

Growth

- Function: NOMINAL
- Purpose: To convert from effective to nominal rate
- Parameters:
  - =nominal(effect_rate,npery)
  - effect_rate is the effective rate
  - npery is the number of periods

Growth

- Converting between compounding conventions
  - The EFFECT and NOMINAL functions can be used to convert between compounding terms.
  - Say an investment earns10% nominal and interest is paid twice yearly. What nominal rate is that equivalent to if interest is paid monthly?

Growth

- Converting between compounding conventions (cont)
  - Step 1. Convert the original nominal rate to effective.
  - = EFFECT(10%, 2) = 10.25%
  - Step 2. Convert the effective rate back to nominal using the new monthly basis
  - = NOMINAL(10.25%, 12) = 9.798%

Simple interest

- An investment P earns simple interest at the rate of i% per year for t years. The investment will then be worth F.
- The relationships between P, F, i & t are:

$$F = P(1+i*t) \qquad i = \frac{1}{t}\left[\frac{F}{P}-1\right]$$

$$P = \frac{F}{1+i*t} \qquad t = \frac{1}{i}\left[\frac{F}{P}-1\right]$$

Annually compounding interest

- An investment P earns annually compounding interest at the rate of i% per year for t years. The investment will then be worth F.
- The relationships between P, F, i & t are:

$$F = P(1+i)^t \qquad i = \left(\frac{F}{P}\right)^{1/t} - 1$$

$$P = \frac{F}{(1+i)^t} \qquad t = \frac{\ln(F/P)}{\ln(1+i)}$$

Nominal compounding interest

- An investment P earns interest m times per year for t years at which time it will be worth F. The nominal yearly rate is i%. The relationships between P, F, i & t are:

$$F = P\left(1+\frac{i}{m}\right)^{mt} \qquad i = m\left[\left(\frac{F}{P}\right)^{1/(mt)} - 1\right]$$

$$P = \frac{F}{\left(1+\frac{i}{m}\right)^{mt}} \qquad t = \frac{\ln(F/P)}{m.\ln\left(1+\frac{i}{m}\right)}$$

Continuous interest

- An investment P earns continuously compounding interest at the rate of i% per year for t years. The investment will then be worth F.
- The relationships between P, F, i & t are:

$$F = Pe^{it} \qquad i = \frac{1}{t}\ln\left(\frac{F}{P}\right)$$

$$P = Fe^{-it} \qquad t = \frac{1}{i}\ln\left(\frac{F}{P}\right)$$

Discount factors

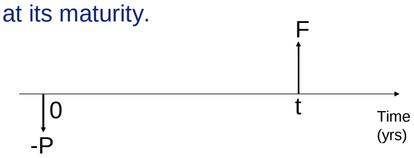
- Discount factors relate the present and future values of a cash flow.
- Let P be the present value, F be the future value and DF be the discount factor.
- Then the relationships between P, F and DF are: $P = DF * F$

$$F = \frac{P}{DF}$$

$$DF = \frac{P}{F}$$

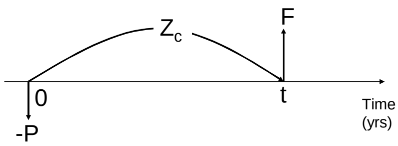
Zero coupon rates

- The term “Zero coupon” refers to a zero coupon bond.
- A zero coupon bond is sold at a discount to its face value and there are no intervening cash flows between that time at its maturity.



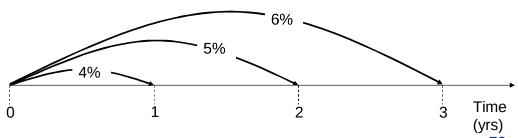
Zero coupon rates

- The zero coupon rate – sometimes called the spot rate – is the interest rate that applies from the present to the time of the future cash flow.



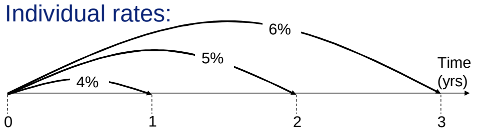
Zero coupon curve

- The zero coupon curve is a set of zero coupon rates each of which relates to a different future time.
- Consider the zero coupon rates below.
- The corresponding “curve” is shown next.

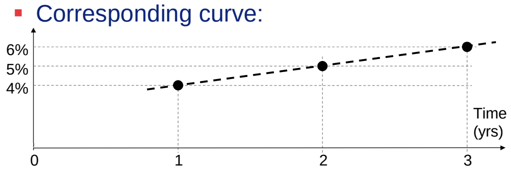


Zero coupon curve

- Individual rates:

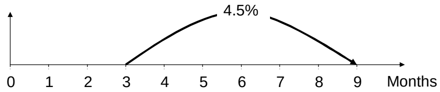


- Corresponding curve:



Forward rates

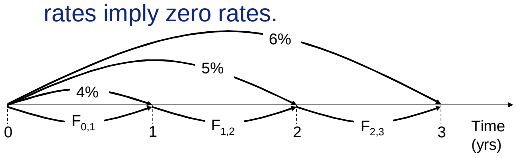
- A forward rate is a rate from some future time to a later time. A forward rate might be for 3 months in the future to 6 months after that:



- This forward rate agreement is a committment that in three months the rate that will apply for the following six months will be 4.5%.

Forward rates relationship to zero rates

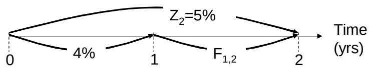
- Zero rates imply forward rates and forward rates imply zero rates.



- What forward rates $F_{0,1}$, $F_{1,2}$ and $F_{2,3}$ are implied by the zero rates above?
- $F_{0,1}$ is the easiest to answer: It is 4.0%.

Forward rates relationship to zero rates

- If we invest for one year at $F_{0,1}$ and then another year at $F_{1,2}$ then we should get the same result as investing for two years at Z_2 (5%).



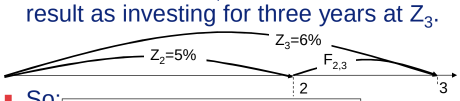
- So: $(1+F_{0,1})(1+F_{1,2}) = (1+Z_2)^2$

$$1+F_{1,2} = \frac{(1+Z_2)^2}{(1+F_{0,1})}$$

$$F_{1,2} = \frac{(1+Z_2)^2}{(1+F_{0,1})} - 1 = \frac{1.05^2}{1.04} - 1 = 6.01\%$$

Forward rates relationship to zero rates

- Investing for two years at Z_2 and then for another year at $F_{2,3}$ should give the same result as investing for three years at Z_3.



- So:

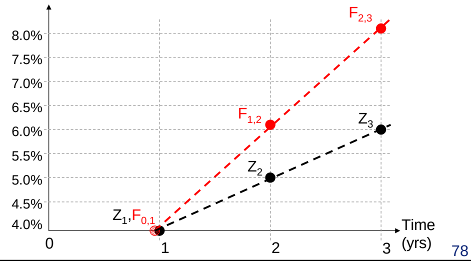
$$(1+Z_2)^2(1+F_{2,3})=(1+Z_3)^3$$

$$1+F_{2,3}=\frac{(1+Z_3)^3}{(1+Z_2)^2}$$

$$F_{2,3}=\frac{(1+Z_3)^3}{(1+Z_2)^2}-1=\frac{1.06^3}{1.05^2}-1=8.03\%$$

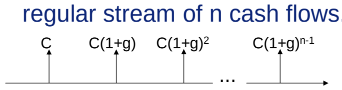
Forward rates relationship to zero rates

- The yield curves on zero and 1-year forward bases are therefore these:



Annuities

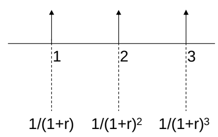
- Consider an annuity composed of a regular stream of n cash flows.



- The first cash flow C is received in one period's time and the next cash flow in two periods. And so on.
- The cash flows grow by a factor of g from one period to the next. g may be zero – in which cash flows are constant.

Annuities

- Assuming a "flat" discount rate of r (annual effective) the k^{th} cash flow is discounted by a factor of $1/(1+r)^k$
- Cash flow:
- Discount factor:



Annuities

- The present value of these annuities is

$$PV = C\left[\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right]$$

- The equation simplifies in a number of cases.

Annuities

- Zero growth. In that case g = 0.

$$PV = \frac{C}{r}\left[1-\left(\frac{1}{1+r}\right)^n\right]$$

- Growth rate equals discount rate. r = g

$$PV = n.C$$

- Perpetual annuity. n →∞

$$PV = \frac{C}{r-g}$$

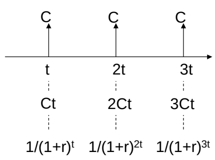
Annuities

- Perpetual annuity, zero growth. n →∞, g=0

$$PV = \frac{C}{r}$$

Time-weighted cash flows

- Formulae relating to time-weighted cash flows have application in various areas including duration and interest rate sensitivity.
- Cash flow
- Cash flow x time
- Discount factor



Time-weighted cash flows

- Consider n cashflows that occur at intervals of t and where the k^{th} cash flow is C.k.t.
- The first cash flow occurs at time t.
- The annual effective discount rate is r.
- The present value of the cashflows is:

$$\frac{Ct}{(a-1)^2}\left[na^{n+2}-(n+1)a^{n+1}+a\right]$$

- where $a=\frac{1}{(1+r)^t}$

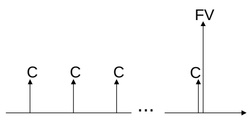
Time-weighted cash flows

- If r is zero then the present value is

$$\frac{Ctn(n+1)}{2}$$

Bonds

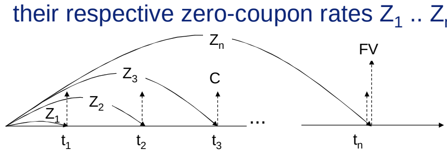
- Bond cash flows looks like the following



- Coupons C are paid regularly. At maturity the principal or face value FV is paid.

Bonds

- The present value of the bond can be obtained by discounting the cash flows at their respective zero-coupon rates Z_1 .. Z_n.

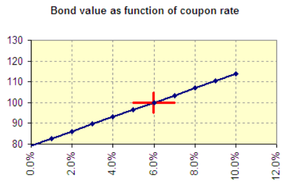


$$PV=\sum_{i=1}^{n}\frac{C}{(1+Z_i)^{t_i}}+\frac{FV}{(1+Z_n)^{t_n}}$$

- Zero rates above are annual effective.

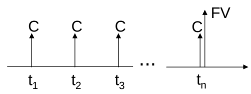
Bonds

- Because C appears only in the numerator of the formula on the preceding page the bond's present value is a linear (and increasing) function of the coupon rate C.



Bonds

- The yield to maturity Y relates the present value of the bond PV to its future cash flows in the scenario of a flat discount curve.



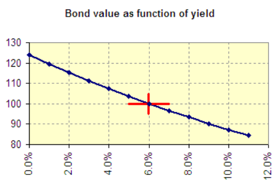
- Y, C, PV and FV are related in this way

$$PV = \sum_{i=1}^{n} \frac{C}{(1+Y)^{t_i}} + \frac{FV}{(1+Y)^{t_n}}$$

- The equation above assumes Y is expressed in annual effective terms.

Bonds

- Because Y appears in the denominator of the formula on the preceding page the bond value is a non-linear (and decreasing) function of the yield.



Bonds – accrued interest

- The bond's total price is obtained by discounting its cash flows (coupons plus principal) to their present values and summing the results.
- The total price can be separated into two components:
  - Accrued interest
  - Capital price
- Total price = accrued interest + capital price

Bonds – accrued interest

- Accrued interest is obtained by pro-rata'ing the coupon by the fractional coupon period remaining.
- e.g. if semi-annual coupon payments are $3 per $100 face value and the next coupon is due in two months then the accrued interest per $100 face value is:
- $3 * 4 / 6 = $2

Bonds – interest rate sensitivity

- The sensitivity of a bond's price to changes in yield is influenced by the times at which its cash flows occur.
- The bond's Macauley duration is the "average" time at which its cash flows will occur.
- The bond's interest rate sensitivity is related to its Macauley duration.

Bonds – interest rate sensitivity

- The Macauley duration is given by the following formula:

$$D = \frac{\sum_{i=1}^{n} t_i.DF_i.CF_i}{\sum_{i=1}^{n} DF_i.CF_i}$$

- t_i = time of cash flow i
- CF_i = cash flow i
- DF_i = discount factor i

Bonds – interest rate sensitivity

- A modified duration is obtained from the Macauley duration according to this formula

$$MD = \frac{D}{1 + \frac{y}{m}}$$

- MD = modified duration
- D = Macauley duration
- y = nominal annual yield
- m = coupon payment per year

Bonds – interest rate sensitivity

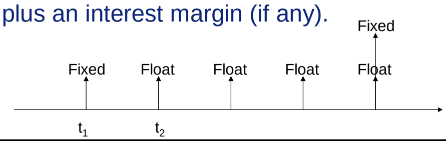
- The interest rate sensitivity is related to the modified duration in this way:

$$\frac{\Delta P}{P} = -\Delta y.MD$$

- $\Delta P/P$ is the fractional change in bond price
- Δy is the change in bond yield
- MD is the modified duration

Floating rate notes

- A floating rate note pays coupons that are periodically “rate-set” to a reference index (e.g. LIBOR, BBSW). An additional interest margin, IM, may be paid.
- At its inception the first coupon will be fixed and set to the then-prevailing reference rate plus an interest margin (if any).

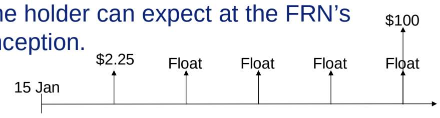


Floating rate notes

- Suppose a FRN with face value $100 is entered into on the 15th of January. The FRN pays quarterly coupons. An interest margin 4% (nominal annual rate) is paid on top of the reference rate.
- The FRN matures on the 15th of April and pays 5 coupons. On maturity the $100 face value is paid as well as the final coupon.
- On the settlement date 15th of January the 90 day reference rate is 5%.

Floating rate notes

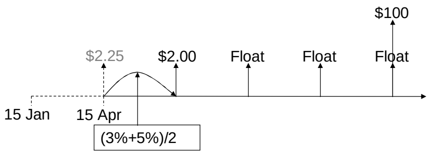
- The diagram below shows the cash flows the holder can expect at the FRN’s inception.



- The first coupon rate will have been set to the then-prevailing 90 day reference spot rate (4%) plus the interest margin (5%). Payments are quarterly on a face value of $100. The first coupon will therefor be 100 * (5%+4%)/4 = $2.25

Floating rate notes

- On the 15th of April the first coupon will be paid out and the next coupon will be rateset. Suppose the then-prevailing reference rate is 3%. Then the second coupon – which will be received in one quarter's time - will be 100*(3%+5%)/4=$2



Floating rate notes

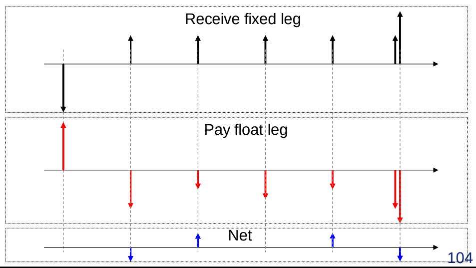
- When determining the present value of the FRN the appropriate discount rate to use should reflect the credit quality of the issuer.
- Suppose the discount rate for a prime bank's debt is D and that the credit qualify of the FRN issuer is such that its debt trades at a margin of M to D then the appropriate rate to use in discounting the FRN cash flows is D+M.

Swaps

- In a fixed-for-floating interest rate swap one party pays fixed and the other pays floating.
- The floating payments are set periodically to a then-prevailing reference rate.
- The notional principal payments cancel out and do not have to be exchanged.
- Typical interest rate swap cash flows are shown on the next slide.

Swaps

- In a fixed-for-floating interest rate swap one party pays fixed and the other pays floating.



IRR

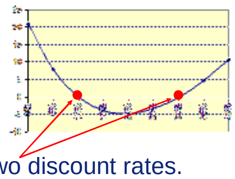
- IRR is the discount rate that sets the NPV of a set of cash flows to zero.
- It is a “yield to maturity”.
- There must be at least one sign change in the cash flows for IRR to be defined.
- Cash flows with more than one change of sign might have more than one IRR.

IRR

- Consider this cash flow:

Period	[yr]	0	1	2
Cashflow	[$ 000's]	699.3007	-1678.32	1000

- The NPV as a function of discount rate looks like this:



- NPV is zero for two discount rates.

IRR

- Spreadsheets use an iterative method to find the IRR: Various discount rates are tried until one is found that gives a zero NPV.
- If no solution is found then an error is returned.

IRR

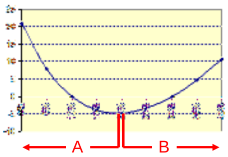
- To calculate IRR use the IRR function..

	A	B	C	D	E
1	Period	[yr]	0	1	2
2	Cashflow	[$ 000's]	699.3007	-1678.32	1000
3					
4	IRR:		=IRR(C2:E2,0%)		

- The first parameter to the IRR function is the set of cash flows whose IRR is to be determined.
- A second, optional, parameter is an initial “guess” which the function uses as the first value in its iterative process.

IRR

- In the example we've been considering an initial guess in region “A” below will cause the IRR function to return the lower solution (10%).
- A guess in region “B” will return 30%.



IRR

- IRR assumes cash flows are spaced at equal intervals.
- The rate returned corresponds to the interval between the periods. e.g. if cash flows are monthly then the IRR will be a monthly rate.

XIRR

- XIRR is like IRR but it uses explicit dates instead of implied ones.
- An example is shown below.

	A	B	C	D
1	15-Jun-2012	15-Sep-2012	15-Dec-2012	15-Mar-2013
2	-300	50	150	160
3				
4	=XIRR(A2:D2,A1:D1)			

- The first parameter is the set of cash flows and the second is the cash flows' dates.
- The annual effective IRR is returned.

Inflation

- An inflation index can be used to track the cumulative effect of year-on-year inflation.
- Suppose we take a certain year as being the “base” year and set the inflation index in that year to 100.
- If inflation in the following year is 3% then the index at the end of that year will be 100*(1+3%) = 103.

Inflation

- If inflation in the following year is 4% then the index at the end of that year will be 103*(1+4%)=107.12
- We can convert nominal dollar amounts in any year into "real" (or inflation-adjusted) figures by multiplying the dollar amount by 100 and dividing by that year's inflation index.

Inflation

- Interest rates which relate to nominal cash flows are termed nominal rates.
- Interest rates which relate to "real" or inflation-adjusted cash flows are termed real rates.
- If inflation is constant year-on-year and the nominal rate curve is flat then there is a simple relationship between real and nominal rates and inflation.

Inflation

- The value today of a nominal cash CF_n flow t years in the future and where the nominal discount rate is r_n is:

$$PV = \frac{CF_n}{(1+r_n)^t}$$

- The value today of a real cash CF_r flow t years in the future and where the real discount rate is r_r is:

$$PV = \frac{CF_r}{(1+r_r)^t}$$

Inflation

- If the constant inflation rate is i then

$$PV = \frac{CF_r}{(1+r_r)^t} = \frac{1}{(1+i)^t} \cdot \frac{CF_n}{(1+r_r)^t}$$

- The present value on real and inflation-adjusted basis must be the same:

$$\frac{1}{(1+i)^t} \cdot \frac{CF_n}{(1+r_r)^t} = \frac{CF_n}{(1+r_n)^t}$$

Inflation

- And so

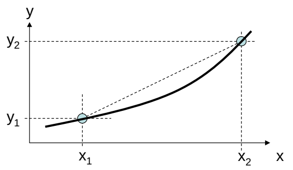
$$1 + r_n = (1 + r_r).(1 + i)$$

- This leads to the following relationship:

$$r_r = \frac{r_n}{1 + i}$$

Sensitivity and gradient

- Suppose y is a function of x

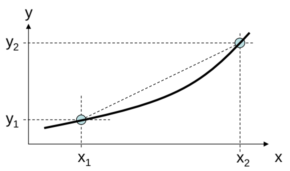


- The sensitivity of y to x (or y's gradient with respect to x) is found by dividing the change in y by the change in x

Sensitivity and gradient

- The gradient is approximately

$$\frac{y_2 - y_1}{x_2 - x_1}$$



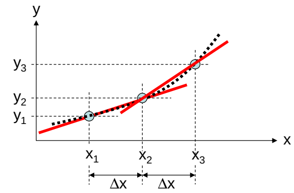
Sensitivity and gradient

- Following is a list of some common functions and their gradients or slopes.

Function y=f(x)	*Gradient y'=f'(x)*
$y=e^{ax}$	$y'=ae^{ax}$
$y=x^n$	$y'=nx^{n-1}$
$y=a^x$	$y'=\ln(a).a^x$
y=ln(x)	y'=1/x
y=f(x).g(x)	y'=f'(x).g(x)+f(x).g'(x)
y=f(g(x))	y'=f'(g(x)).g'(x)

Curvature

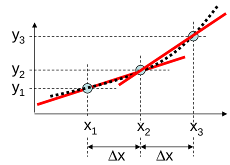
- Curvature is the gradient of the gradient.
- It is a measure of how quickly the slope of y(x) varies with x.



Curvature

- The curvature is:

$$\frac{\frac{y_3 - y_2}{\Delta x} - \frac{y_2 - y_1}{\Delta x}}{\Delta x} = \frac{y_3 - 2y_2 + y_1}{\Delta x^2}$$



Mean of a frequency distribution

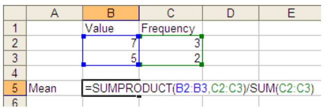
- Assume that – in a series of results - 7 occurs 3 times
- A value of 5 occurs 2 times.
- What is the mean value?

$$\overline{V} = \frac{7*3+5*2}{3+2} = \frac{31}{5} = 6.2$$

- The spreadsheet function SUMPRODUCT is useful for performing calculations like this.

Mean of a frequency distribution

- The mean is found by dividing the SUMPRODUCT of the values and frequencies by the SUM of the number of frequencies.



	A	B	C	D	E
1		Value	Frequency		
2		7	3		
3		5	2		
4					
5	Mean	=SUMPRODUCT(B2:B3,C2:C3)/SUM(C2:C3)			
6					

Mean of a frequency distribution

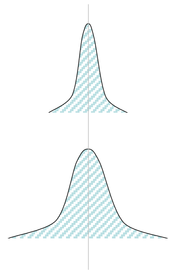
- If the frequencies are expressed on a percentage basis then they will add to 1. In this case the calculation of mean is simplified.

	A	B	C	D
1		Value	Frequency [%]	
2		7	60%	
3		5	40%	
4				
5	Mean	=SUMPRODUCT(B2:B3,C2:C3)		

- There is no need to divide by the sum of the frequencies since the sum is 1.

Standard deviation

- Is a measure of the "spread" of a distribution.
- Both the distributions to the right have the same mean. But the lower distribution has a greater standard deviation.



Standard deviation

- Standard deviation can be calculated on one of two bases: 1) "Sample", 2) "Population".
- Population basis means you have the entire set of data (e.g. a set of asset prices in five months's time assuming an asset can go up or down by a certain amount in each month).
- The spreadsheet function STDEVP returns the population standard deviation.

Standard deviation

- The formula for calculating the standard deviation σ of a population which contains n items x_1 .. x_n is this:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}$$

- where μ is the mean of x_1 .. x_n

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Standard deviation

- Standard deviation on a sample basis assumes you don't have the entire population but only have a sample of the set.
- For example you might have a sample of historical returns from which you're estimating the underlying volatility.

Standard deviation

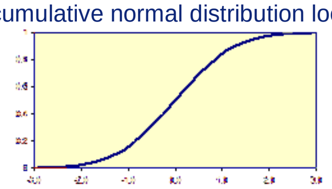
- The function for sample standard deviation is this:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu)^2}$$

- The spreadsheet function to return the sample standard deviation is STDEV

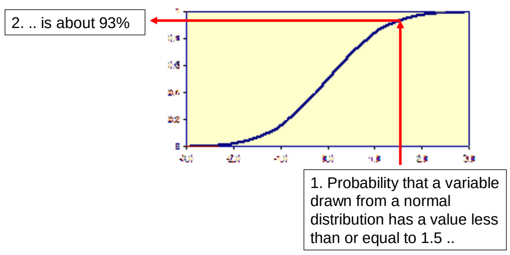
Cumulative normal distribution function

- The function NORMSDIST returns the cumulative normal distribution for the case when the mean is zero and the standard deviation is 1.
- The cumulative normal distribution looks like this:



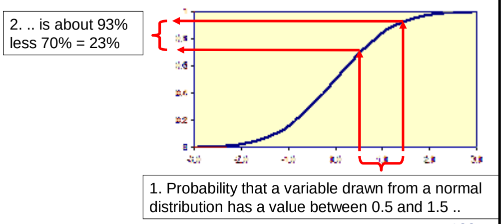
Cumulative normal distribution function

- The function can be used this way:



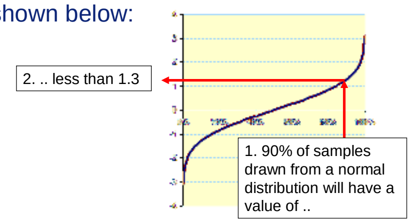
Cumulative normal distribution function

- To find the probability of x being between x_1 and x_2 we find the difference between the cumulative distribution for x_1 and for x_2:



Inverse cumulative normal distribution function

- The NORMSINV function returns the inverse of the cumulative normal distribution. The function has the profile shown below:



NORMDIST and NORMINV functions

- The NORMSDIST and NORMSINV functions assume the normal distribution has a mean of 0 and standard deviation of 1.
- The NORMDIST and NORMINV functions are more general versions of their counterparts and allow you to work with other means and standard deviations.

NORMDIST function

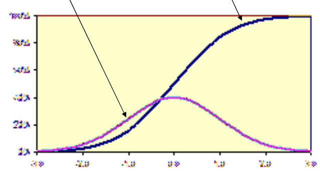
- The NORMSDIST function takes four arguments
- NORMDIST(x, mean, sd, cumulative)
- The first argument is the value for which you want the distribution. e.g. if the first value is 0.5 then the probability will be returned that the normally distributed random variable is less than (or equal to) 0.5.

NORMDIST function

- The second argument is mean of the distribution.
- The third argument is the standard deviation.
- The fourth argument is a flag that is TRUE or FALSE.
  - TRUE = return the cumulative probability
  - FALSE = return the probability "mass" function

NORMDIST function

- Cumulative function
- "Mass" function



NORMDIST function

- How are the cumulative and mass functions related?
  - The mass function is the slope or gradient of the cumulative function.
  - The cumulative function is the integral or "running total" of the mass function.
  - Say the Mass function for a value of x is 0.35. Then the probability that a normally distributed random variable will be between x and x + δx will be 0.35 * δx (where δx is a small number).

NORMINV function

- The NORMINV function is a generalised version of the NORMSINV function.
- It additionally lets you specify a mean and standard deviation (the NORMSINV assumes the mean is 0 and the standard deviation is 1).
- NORMINV(probability, mean, standard deviation)

NORMINV function

- The first parameter is the probability. e.g if the first parameter is 0.95 then the function will return the number of standard deviations above the mean below which 95% of random variables will occur.
- The second parameter is the mean.
- The third parameter is the standard deviation.

Generating random variables drawn from a rectangular distribution

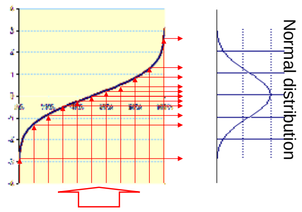
- To generate numbers drawn from a rectangular distribution use the RAND() function.
- This generates random numbers between 0 and 1.
- Each number between 0 and 1 is equally likely to occur.

Generating random variables drawn from a normal distribution

- To generate normally distributed numbers with a mean of 0 and standard deviation of 1 use NORMSINV(RAND())
- In this technique rectangularly distributed numbers are passed to the NORMSINV function. The result is random numbers drawn from the normal distribution.

Generating random variables drawn from a normal distribution

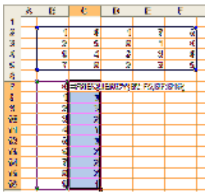
- The NORMSINV function causes the outcomes from rectangularly distributed numbers to "bunch" in a normal fashion ..



Rectangular distribution

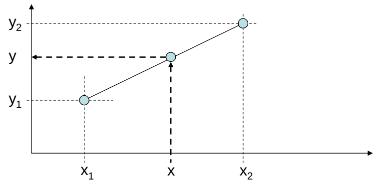
Tabulating a frequency distribution

- The FREQUENCY function can be used to tabulate the frequency with which events fall into defined intervals.
- The first argument defines the intervals
- The second argument defines the data to be analysed.

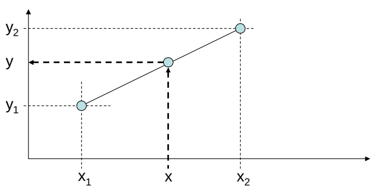


Interpolation - linear

- Involves “filling in the gaps” between discrete data points.
- Linear interpolation. Given x_1, y_1, x_2, y_2 and x – we want to find y



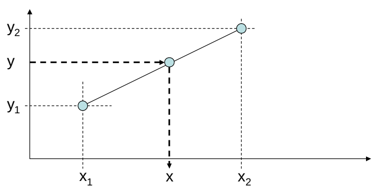
Interpolation - linear



- Formula for y is:

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

Interpolation - linear

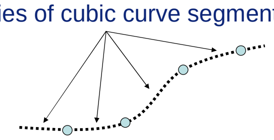


- Given y we can find x:

$$x = x_1 + \frac{(y - y_1)(x_2 - x_1)}{(y_2 - y_1)}$$

Interpolation – cubic spline

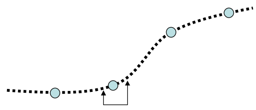
- Cubic spline interpolation involves fitting a series of cubic curve segments.



- A cubic curve has an equation like this:
  - $y = ax^3 + bx^2 + cx + d$

Interpolation – cubic spline

- At each data point (other than first and last) two curve segments meet.
- The curve segments' values, gradients and curvatures match at the data point.

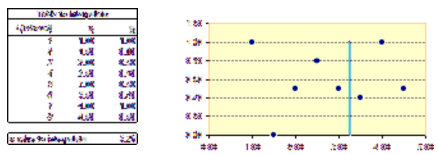


Interpolation – cubic spline

- Need to impose "boundary conditions" at the leftmost and rightmost data points.
- Boundary conditions could be that the curvature be zero at the leftmost and rightmost points.
- This is called a "natural" spline.
- We'll show the method with an example ..

Interpolation – cubic spline

- Data we will interpolate is shown below in tabular and graphical form



- We want to find the interpolated value corresponding to x = 3.25.

Interpolation – cubic spline

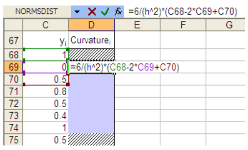
- Step 1 – Calculate "h" – the interval between any two x values (x values are evenly spaced).
- In our example x increases in steps of 0.50 so h is 0.50.
- Step 2 – Calculate which interval x is in. In our example x is 3.25 and is in interval 5 (counting the first interval as 5).

Interpolation – cubic spline

- Step 3 – Calculate x's offset within its interval. Interval 5 spans x values from 3.00 to 3.50 so x's offset within the interval is (3.25 – 3.00) = 0.25. Call this offset Δx
- Step 4 – Calculate "curvatures". We omit the first and last intervals and calculate a scaled curvature for the remaining points.

Interpolation – cubic spline

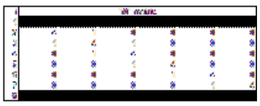
- Step 4 (cont) – The snapshot below shows how the curvature C_i for the i_{th} data point is defined in terms h and the data points at i-1, i and i+1.



- $C_i = 6/(h^2)*(y_{i-1}-2y_i+y_{i+1})$

Interpolation – cubic spline

- Step 5. Generate an "M" matrix.



- The matrix contains rows of this form:
  - .....0 0 0 1 4 1 0 0 0 ....
  - On each successive rows the pattern shifts one to the right.
  - The top row starts with 4 1 0 0 .. and the last row ends with ... 0 0 1 4

Interpolation – cubic spline

- Step 6. Multiply the inverse of the M matrix by the curvatures to generate a vector whose i^{th} element is M_i

- To do this use the MINVERSE function to find the matrix inverse and use MMULT to do the matrix multiplication.

Interpolation – cubic spline

- Step 6 (cont). The formula will return an array of values. Array formulae need to be entered in a special way.
  - i) Select all of the cells and then type the formula.

  - ii) Press CTRL + SHIFT + ENTER whilst the cursor is in the formula bar:

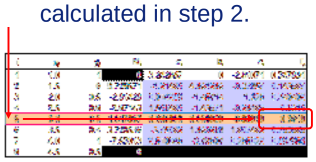
Interpolation – cubic spline

- Step 7. Calculate a, b and c coefficients and interpolated y values I_i for all of the intervals i from 1 through to the second-last.
  - $a_i = (M_{i+1}-M_i)/(6*h)$
  - $b_i = M_i / 2$
  - $c_i = (y_{i+1}-y_i)/h-(M_{i+1}+2*M_i)*h/6$

    $I_i = a_i*dX^3+b_i*dX^2+c_i*dX+y_i$

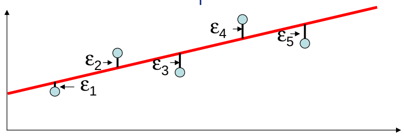
Interpolation – cubic spline

- Step 8. This is the last step. Choose the interpolated value from the interval calculated in step 2.



Curve fitting - TREND

- The TREND function fits a “least-squares” line to a set of data points.
- Given a set of data points (represented by the dots below) TREND gives the line that minimises the sum of the squares of the differerence terms ε_i.



Curve fitting - TREND

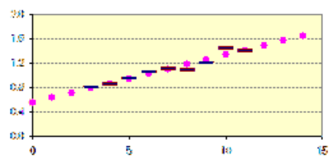
- The TREND function takes three arguments.
- The first is the “Y” values of the data points.
- The second is the “X” values of the data points.
- The third is a set of x values at which the interpolation will be done.

Curve fitting - TREND

- We’ll study an example of using TREND.
- B6:B14 are the y values
- A6:A14 are the x values
- A3 is the x value we want to “interpolate”.
- Following slide shows the result graphically

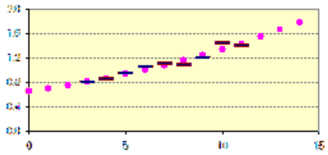
Curve fitting - TREND

- The horizontal markers show the original data points.
- The dots show the points generated by the TREND function



Curve fitting - GROWTH

- The GROWTH function is like TREND but it fits an exponentially growing curve rather than a straight line.
- The result of using the GROWTH function with the same data set as before is shown next.

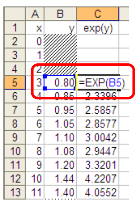


Curve fitting – Other functions

- Suppose we want to fit a function F to a curve. We can use this method:
  - Transform each y data point value by F's inverse function G.
  - G and F are related this way: G(F(y)) = y
  - If, for example, F(y) is ln(y) then the inverse function is G(y) = exp(y).
  - Use the linear TREND function on G(y) to generate interpolated values.
  - Re-transform the interpolated values with the F function.

Curve fitting – Other functions

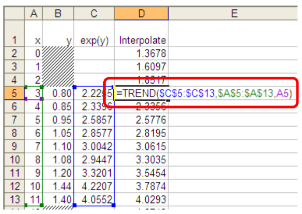
- This is how the method can be applied to fit a logarithmic curve to a set of data points.
- Step 1. Generate a set of transformed y values by using the exp function (exp is the inverse function to the natural log function.



	A	B	C
1	x	y	exp(y)
2	0		
3	1		
4	2		
5	3	0.80	=EXP(B5)
6	4	0.85	2.3396
7	5	0.95	2.5857
8	6	1.05	2.8577
9	7	1.10	3.0042
10	8	1.08	2.9447
11	9	1.20	3.3201
12	10	1.44	4.2207
13	11	1.40	4.0552

Curve fitting – Other functions

- Step 2. Use the TREND function on the transformed y values.



	A	B	C	D	E
1	x	y	exp(y)	Interpolate	
2	0			1.3678	
3	1			1.6097	
4	2			1.8517	
5	3	0.80	2.2255	=TREND($C$5:$C$13,$A$5:$A$13,A5)	
6	4	0.85	2.3396	2.3356	
7	5	0.95	2.5857	2.5776	
8	6	1.05	2.8577	2.8195	
9	7	1.10	3.0042	3.0615	
10	8	1.08	2.9447	3.3035	
11	9	1.20	3.3201	3.5454	
12	10	1.44	4.2207	3.7874	
13	11	1.40	4.0552	4.0293	

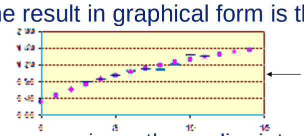
Curve fitting – Other functions

- Step 3. Use the logarithmic (ln) function on the interpolated, transformed y values.

	A	B	C	D	E
1	x	y	exp(y)	Interpolate	ln(Interpolate)
2	0			1.3678	0.3132
3	1			1.6097	0.4761
4	2			1.8517	0.6161
5	3	0.80	2.2255	2.0936	=LN(D5)
6	4	0.85	2.3396	2.3356	0.8483
7	5	0.95	2.5857	2.5776	0.9468
8	6	1.05	2.8577	2.8195	1.0366
9	7	1.10	3.0042	3.0615	1.1189
10	8	1.08	2.9447	3.3035	1.1950
11	9	1.20	3.3201	3.5454	1.2657
12	10	1.44	4.2207	3.7874	1.3317
13	11	1.40	4.0552	4.0293	1.3936

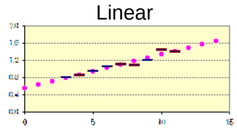
Curve fitting – Other functions

- The result in graphical form is this

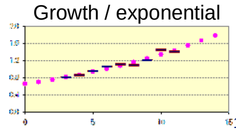


- By comparison, the earlier interpolations performed on other bases were:

Linear

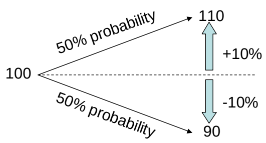


Growth / exponential



Asset evolution

- Suppose a $100 asset can increase or decrease in value by 10% over a period.
- Suppose each possibility is equally likely.
- In one period's time:



Asset evolution

- The expected or mean return is the sum of the probability-weighted individual returns
- Expected return = 0.5 * 10% + 0.5 * (-10%) =0%
- The standard deviation of the returns is:

$$\sqrt{\frac{(10\%-0\%)^2+(-10\%-0\%)^2}{2}}$$

$$=\sqrt{\frac{2*10\%^2}{2}}=10\%$$

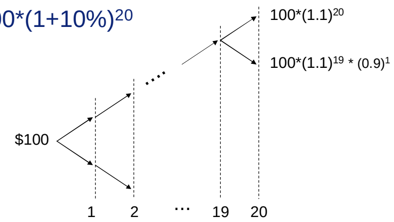
Asset evolution

- Alternatively the standard deviation can be calculated by using the STDEVP function:

	A	B	C	D
1	Price in one period	Probability	Return	
2	110	50%	10%	
3	90	50%	-10%	
4				
5				
6	Standard deviation		=STDEVP(C2 C3)	

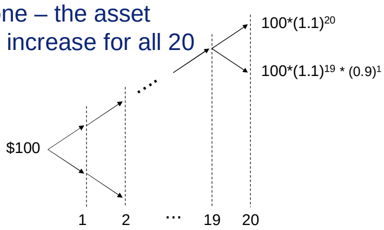
Asset evolution

- Q. What is the largest value the asset could have at the end of 20 periods?
- A. $100*(1+10\%)^{20}$



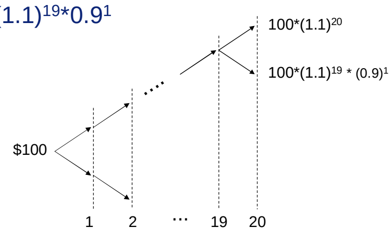
Asset evolution

- Q. How many “paths” lead to the previous result?
- A. Just one – the asset needs to increase for all 20 periods.



Asset evolution

- Q. What is the next-highest value the asset could take?
- A. $100*(1.1)^{19}*0.9^{1}$



Asset evolution

- Q. How many paths lead to the previous result?
- A. The asset has to increase over 19 periods and fall in one period. There are 20 ways this can happen. i.e. The “down” move can be in any of 20 periods.

Asset evolution

- Q. What is the next highest value the asset could take?
- A. $100*(1.1)^{18}*0.9^{2}$
- Q. How many paths lead to that result?
- A. The first “down” move can be in any of 20 periods and the second in any of the remaining 19. This gives 20*19 permutations.

Asset evolution

- A (cont). However, we are “double-counting” and really need the number of combinations of ups and downs rather then the number of permutations.
- So we need, in this case, to divide by two. The number of paths is 20*19/2 = 190.
- In general the number of paths that contain k “up” moves in n periods is:

$$\frac{n!}{k!(n-k)!}$$

Asset evolution

- We can use the BINOMDIST function to calculate the probability of reaching any particular asset value in 20 periods.
- The probability of having k up moves in n periods is calculated this way:
- =BINOMDIST(k, n, p, FALSE)
- where p is the probability of an up move

Asset evolution

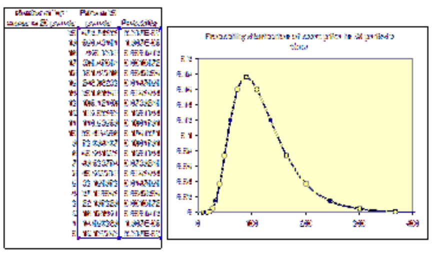
- The fourth parameter in the BINOMDIST function determines whether or not it returns a cumulative probability.
- If the fourth parameter is TRUE then a cumulative result is given (i.e. probability that k <u>or fewer</u> up moves happen in n periods)
- If the fourth parameter is FALSE then the non-cumulative result is given.

Asset evolution

- Using the BINOMDIST function we can calculate the probability of each possible asset price in 20 periods.
- The prices and probabilities can be plotted on an "XY" chart
- Prices are shown on the x axis
- Probabilities are shown on the y axis.
- The result is shown on the next slide.

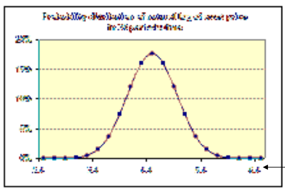
Asset evolution

- The probability distribution has a characteristic "log-normal" shape.



Asset evolution

- If the probability distribution is log-normal then the probability distribution of the log of the asset price is normal.



Log of asset price

Asset evolution

- For a log-normal distribution the mean (μ) and standard deviation (sd) of the log of the future asset price at time T years is defined by:

$$\mu = \ln(S) + \left(g - \frac{\sigma^2}{2}\right)T,$$

$$sd = \sigma.\sqrt{T}$$

- where g is the expected short-term growth rate and σ is the standard deviation (annualised) of the asset's return.

Asset evolution

- The preceding equations can be used to calculate confidence intervals on the future value of a log-normally distributed asset.
- Alternatively, the following formulae can be used.

$$S_{\max} = Se^{(g-\sigma^2/2)T + n\sigma\sqrt{T}}$$

$$S_{\min} = Se^{(g-\sigma^2/2)T - n\sigma\sqrt{T}}$$

Asset evolution

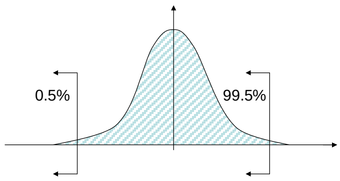
- In the preceding equations S_{max} is the upper confidence interval on the asset price, S_{min} is the lower, g is the expected short-term growth rate, σ is the annualised return volatility and n is the standard deviation around the expected price.

$$S_{\max} = Se^{(g-\sigma^2/2)T + n\sigma\sqrt{T}}$$

$$S_{\min} = Se^{(g-\sigma^2/2)T - n\sigma\sqrt{T}}$$

Asset evolution

- Suppose we want 99% confidence intervals on the price. What is n?
- 99.5% of prices will be below the upper limit and 0.5% below the lower limit.



Asset evolution

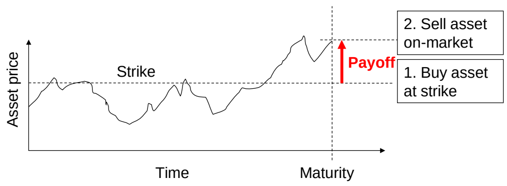
- How many standard deviations above the mean will capture 99.5% of outcomes?
- Calculate this by using the inverse normal cumulative distribution function NORMSINV.
- NORMSINV(0.995) is 2.576
- i.e. 99.5% of outcomes will be less than 2.576 standard deviations above the mean. So n is 2.576.

Options

- A call option gives its holder the right but not the obligation to buy an asset at an agreed price at (or before) a maturity date. The agreed price is the strike or exercise price.
- The payoff of an option is its value when exercised. The payoff is the difference between the strike price and the then-prevailing market price of the asset, or zero, whichever is greater.

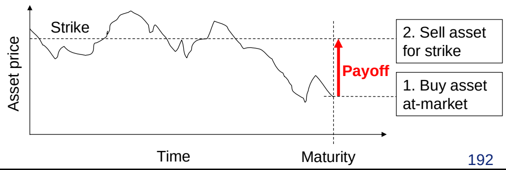
Options

- The diagram below shows the payoff of a call option that expires in-the-money.



Options

- A put option gives its holder the right to sell an asset at an agreed price at (or before) a maturity date.
- The diagram below shows the payoff of a put option that expires in-the-money.

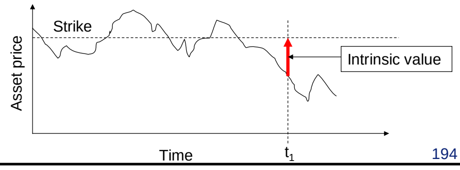


Options

- The holder of an American option can exercise at any time up to and including maturity.
- The holder of a European option can exercise only at maturity.

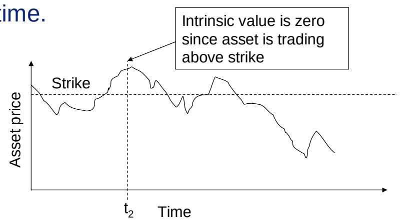
Options

- The intrinsic value of an option is the payoff that would be achieved if the option were exercised immediately.
- The diagram below shows the intrinsic value at time t_1 of a put option.



Options

- The intrinsic value at time t_2 of the put option illustrated below is zero because the option would not be exercised at that time.



Options

- An option is at-the-money (ATM) if the price of the underlying asset equals the strike price.
- An option is in-the-money (ITM) if the intrinsic value is greater than zero.
- An option is out-of-the-money (OTM) if the intrinsic value is zero.

Options

- The option “greeks” are measures of an option’s sensitivity to the factors that affect its value.
- The delta is the sensitivity of the option value to the underlying asset’s price.
- The delta is the change in option value divided by the change in option price.

Options

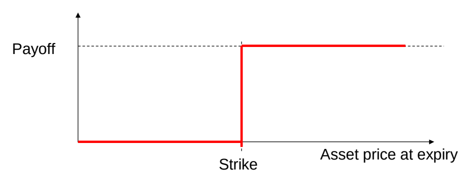
- The theta is the sensitivity to time.
- The theta is the change in option value divided by the change in time.
- The gamma of an option is the sensitivity of its delta to changes in the underlying asset price.
- The gamma is the change in delta divided by the change in the underlying asset price.

Options

- The vega of an option is the sensitivity of its value to changes in the volatility of the underlying asset. The vega is the change in option value divided by the change in volatility.
- The rho of an option is its sensitivity to changes in the interest rate.
- The rho is the change in option value divided by the change in interest rate.

Digital Options

- A digital option pays a fixed amount if it expires in-the-money and zero otherwise.
- The payoff profile of a call digital option is shown below.

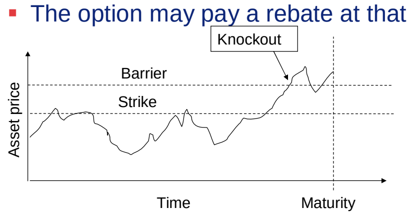


Barrier Options

- Barrier options can “knock in” or “knock out”.
- A barrier option knocks out if the underlying asset trades through a specified barrier.
- An up-and-out option knocks out if the asset rises above the barrier.
- A down-and-out option knocks out if the asset falls below the barrier.

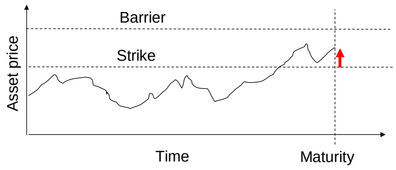
Barrier Options

- The up-and-out call barrier option illustrated below knocks out – It ceases to exist at the knockout time.
- The option may pay a rebate at that time.



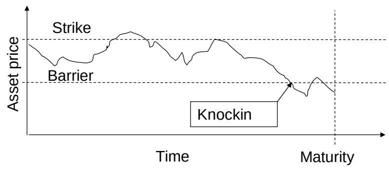
Barrier Options

- The up-and-out call barrier option illustrated below doesn’t knock out and gives the same payoff at expiry as the equivalent vanilla call option.



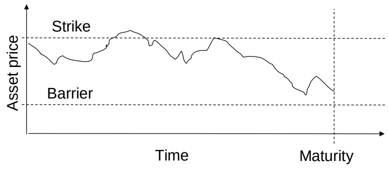
Barrier Options

- The down-and-in put barrier option illustrated below knocks in and becomes a vanilla put option at that time.



Barrier Options

- The down-and-in put barrier option illustrated below never knocks in and will give no payoff. However, if a rebate is specified the rebate will be paid at maturity in the case there is no knockin.

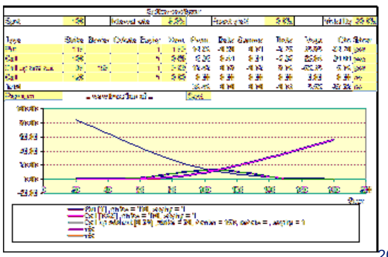


Option analyser

- An option analyser is included in one of the option exercise sheets.
- The option analyser lets you study the behaviour of various types of options.
- You can chart and tabulate the greeks both for individual options and for portfolios of options.

Option analyser

- The main features of the analyser are described next. The analyser looks like this:



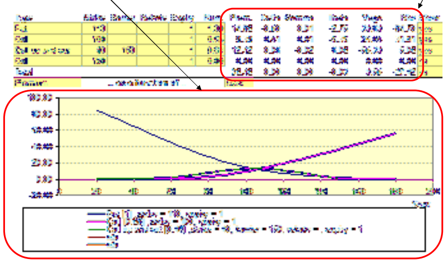
Option analyser

- You can change any cell that is yellow ..

Spot	100		Interest rate		5.0%	Asset yield		0.0%		Volatility	30.0%
Type	Strike	Barrier	Rebate	Expiry	Num.	Prem.	Delta	Gamma	Theta	Vega	Rho	Show
Put	110			1	1.00	14.66	-0.50	0.01	-2.75	39.89	-64.70	yes
Call	100			1	0.66	9.39	0.41	0.01	-5.35	25.04	31.81	yes
Call up and out	80	150		1	0.89	12.42	0.08	-0.02	8.06	-55.39	5.06	yes
Call	120			1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	no
Total						36.46	0.00	0.00	-0.03	9.55	-27.82	no
Premium		.. as a function of ..				Spot						

Option analyser

- After you change a yellow cell the “greeks” and charts will automatically update.



Option analyser

- Option types allowed by the analyser are:
  – Call
  – Put
  – Call digital
  – Put digital
  – Barriers
    - Call / Put
    - Up / down
    - Out / in
    - (e.g. Up-and-out call, down-and-in put)

Option analyser

- Option types (cont)
- All options are European
- In addition two non-option types are allowed
  – Asset
  – Cash
- The following slides give more information about each option type.

Option analyser

- Call / Put
- Required parameters are:
  – Strike
  – Expiry (in years)
  – Spot
  – Asset yield (% p.a. continuously compounding)
  – Volatility (% p.a.)
  – Interest rate (% p.a. continuously compounding)
  – Number held

Option analyser

- Call digital / Put digital
- These pay $1 if the option expires in the money and zero otherwise.
- Required parameters are the same as for vanilla call / put options.

Option analyser

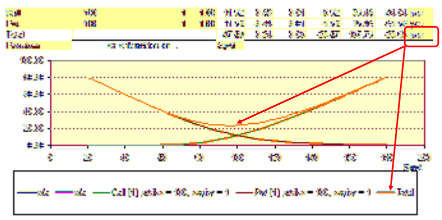
- Barrier options
- These comprise call up and out, call up and in, call down and out, call down and in, put up and out, put up and in, put down and out and put down and in.
- Required parameters are the same as for the vanilla call / put plus two extra:
  - Barrier level
  - Rebate

Option analyser

- Barrier options (cont)
- For “out” options the rebate is paid at the time of knockout.
- For “in” options the rebate is paid at maturity if the option has not knocked in.

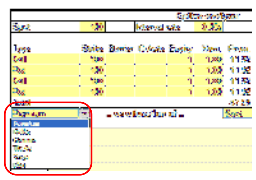
Option analyser

- The “Total” flag can be “yes” or “no”. If it is “yes” then the individual results are totalled and an additional total curve is shown.



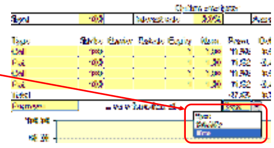
Option analyser

- Greeks that can be charted are:
  - Premium
  - Delta
  - Gamma
  - Theta
  - Vega
  - Rho



Option analyser

- The parameter that is varied in order to generate charts can be spot, volatility or time.
  - Spot
  - Volatility
  - Time



- If Spot or Volatility is chosen then the chart will show results for spot or volatility varying by +/- 80% from its current value.

Option analyser

- If Time is chosen then the chart will show results from times from 0 to the nearest option's maturity.

Analytic formulae

- The value of European call and put options on a yield-paying asset is given by the following Black-Scholes formulae:

$$c = Se^{-qT}N(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

- where

$$d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Analytic formulae

- In the previous formula:
  - c = call option price
  - p = put option price
  - S = spot price of the asset
  - T = time to expiry (in years)
  - X = strike
  - r = risk free interest rate (continuously comp.)
  - q = asset yield (continuously compounding)
  - σ = volatility of the asset's log-return
  - N() = cumulative normal distribution

Analytic formulae

- The deltas of call and put options on yield bearing assets are given by:

$$\Delta_c = e^{-qT} N(d_1)$$

$$\Delta_p = e^{-qT} [N(d_1) - 1]$$

- where q,T & d_1 are as before

Analytic formulae

- The value of an option on a future is given by the following Black futures option formulae:

$$c = e^{-rT}\left[FN(d_1) - XN(d_2)\right]$$

$$p = e^{-rT}\left[XN(-d_2) - FN(-d_1)\right]$$

- where

$$d_1 = \frac{\ln(F/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Analytic formulae

- In the previous formula:
  - c = call option price
  - p = put option price
  - F = spot price of the future
  - T = time to expiry (in years)
  - X = strike
  - σ = volatility of the asset's log-return
  - N() = cumulative normal distribution

Analytic formulae

- The deltas of call and put options on futures are given by:

$$\Delta_c = e^{-rT} N(d_1)$$

$$\Delta_p = e^{-rT} [N(d_1) - 1]$$

- where r,T & d_1 are as defined before

Analytic formulae

- As an example of a more complex analytic formula consider the following options which pay on the maximum or minimum of two assets.
- Call on minimum of two assets
- Payoff = max(min(S_1,S_2)-X,0)

$$c_{\min} = S_1 e^{-q_1 T} M(y_1, -d; -\rho_1) + S_2 e^{-q_2 T} M(y_2, d - \sigma\sqrt{T}; -\rho_2) - Xe^{-rT} M(y_1 - \sigma_1\sqrt{T}, y_2 - \sigma_2\sqrt{T}; \rho)$$

Analytic formulae

- where

$$d = \frac{\ln(S_1/S_2) + (q_2 - q_1 + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$y_1 = \frac{\ln(S_1/X) + (r - q_1 + \sigma_1^2/2)T}{\sigma_1\sqrt{T}}$$

$$y_2 = \frac{\ln(S_2/X) + (r - q_2 + \sigma_2^2/2)T}{\sigma_2\sqrt{T}}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

$$\rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma}, \rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma}$$

Analytic formulae

- and ρ is the correlation between the two assets' returns.
- M is the cumulative bivariate normal distribution.
- There is no intrinsic M function in spreadsheets and a common way of making M available is as a user-defined Visual Basic function.
- Suitable code for a Visual Basic function is shown on the following slide:

Analytic formulae

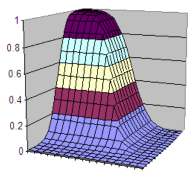
```
Option Explicit
Option Base 1

Public Function BivN(a, b, rho)
    Dim aarray, barray, rho1, rho2, delta, ap, bp, Sum, i, j, Pi, denum

    aarray = Array(0.24840615, 0.39233107, 0.21141819, 0.03324666, 0.00082485334)
    barray = Array(0.10024215, 0.48281397, 1.0609498, 1.7797294, 2.6697604)

    Pi = 3.14159265358979
    ap = a / Sqr(2 * (1 - rho ^ 2))
    bp = b / Sqr(2 * (1 - rho ^ 2))

    If a <= 0 And b <= 0 And rho <= 0 Then
        Sum = 0
        For i = 1 To 5
            For j = 1 To 5
                Sum = Sum + aarray(i) * aarray(j) * f(barray(i), barray(j), ap, bp, rho)
            Next
        Next
        BivN = Sum * Sqr(1 - rho ^ 2) / Pi

    ElseIf a <= 0 And b >= 0 And rho >= 0 Then
        BivN = Norm(a) - BivN(a, -b, -rho)
    ElseIf a >= 0 And b <= 0 And rho >= 0 Then
        BivN = Norm(b) - BivN(-a, b, -rho)
    ElseIf a >= 0 And b >= 0 And rho <= 0 Then
        BivN = Norm(a) + Norm(b) - 1 + BivN(-a, -b, rho)
    ElseIf a * b * rho >= 0 Then
        denum = Sqr(a ^ 2 - 2 * rho * a * b + b ^ 2)
        rho1 = (rho * a - b) * Sgn(a) / denum
        rho2 = (rho * b - a) * Sgn(b) / denum
        delta = (1 - Sgn(a) * Sgn(b)) / 4
        BivN = BivN(a, 0, rho1) + BivN(b, 0, rho2) - delta
    End If
End Function
```

```
Public Function f(x, y, ap, bp, rho)

Dim r
r = ap * (2 * x - ap) + bp * (2 * y - bp) + 2 * rho * (x - ap) * (y - bp)
f = Exp(r)
End Function

Private Function Norm(x)
    Norm = Application.WorksheetFunction.NormSDist(x)
End Function
```

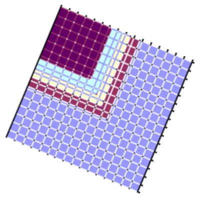

Analytic formulae

- The bivariate normal distribution for a correlation of +1 looks like this:

Perspective view



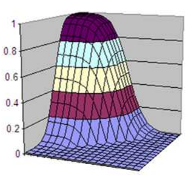
Top view



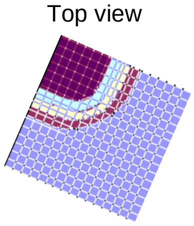
Analytic formulae

- The bivariate normal distribution for a correlation of -1 looks like this:

Perspective view



Top view



Analytic formulae

- Continuing with the Multi-asset option these are the formulae for a call on the maximum of two assets
- Payoff = max(max(S_1,S_2)-X,0)

$$c_{\max} = S_1 e^{-q_1 T} M(y_1, d; \rho_1) +$$
$$S_2 e^{-q_2 T} M(y_2, -d + \sigma\sqrt{T}; \rho_2)$$
$$- Xe^{-rT}\left[1 - M(-y_1 + \sigma_1\sqrt{T}, -y_2 + \sigma_2\sqrt{T}; \rho)\right]$$

Analytic formulae

- Barrier options
- As an example of analytic pricing of barriers the following are formulae for pricing standard barrier options.

$$A = \phi S e^{-qT} N(\phi x_1) - \phi X e^{-rT} N(\phi x_1 - \phi\sigma\sqrt{T})$$
$$B = \phi S e^{-qT} N(\phi x_2) - \phi X e^{-rT} N(\phi x_2 - \phi\sigma\sqrt{T})$$
$$C = \phi S e^{-qT} (H/S)^{2(\mu+1)} N(\eta y_1) - \phi X e^{-rT} (H/S)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T})$$
$$D = \phi S e^{-qT} (H/S)^{2(\mu+1)} N(\eta y_2) - \phi X e^{-rT} (H/S)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})$$
$$E = K e^{-rT} \left[N(\eta x_2 - \eta\sigma\sqrt{T}) - (H/S)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T}) \right]$$
$$F = K\left[(H/S)^{\mu+\lambda} N(\eta z) + (H/S)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T}) \right]$$

Analytic formulae

- where

$$x_1 = \frac{\ln(S/X)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, x_2 = \frac{\ln(S/H)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$$
$$y_1 = \frac{\ln(H^2/(SX))}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, y_2 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}$$
$$z = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}, \mu = \frac{r - q - \sigma^2/2}{\sigma^2}, \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$

- and S = spot, X = strike, H = barrier, K = rebate, r = risk free rate, q = asset yield, T = time to expiry and σ = volatility

Analytic formulae

- The rules for combining A .. F to obtain a barrier option premium are listed next.
- For a down and out call, η=1 ϕ=1
  - If (X>H) then premium = C+E
  - else premium = A-B+D+E
- For an up and in call, η=-1 ϕ=1
  - If (X>H) then premium = A+E
  - else premium = B-C+D+E

Analytic formulae

- For a down and in put, η=1 ϕ=-1
  - If (X>H) then premium = B-C+D+E
  - else premium = A+E
- For an up and in put, η=-1 ϕ=-1
  - If (X>H) then premium = A-B+D+E
  - else premium = C+E
- For a down and out call, η=1 ϕ=1
  - If (X>H) then premium = A-C+F
  - else premium = B-D+F

Analytic formulae

- For a up and out call, η=-1 ϕ=1
  - If (X>H) then premium = F
  - else premium = A-B+C-D+F
- For a down and out put, η=1 ϕ=-1
  - If (X>H) then premium = A-B+C-D+F
  - else premium = F
- For an up and out put, η=-1 ϕ=-1
  - If (X>H) then premium = B-D+F
  - else premium = A-C+F

Analytic formulae

- If an option knocks in then it should no longer be valued as a barrier option but instead a European price should be calculated.

Binomial pricing

- Let S be the spot price of an asset,
- T be the expiry time of the option
- X be the strike
- r be the risk free rate
- q be the asset yield, and
- σ be the volatility

Binomial pricing

- Step 1. Choose n – the number of time intervals between now (time 0) and expiry (time t).
  - Calculate the time interval $\Delta t = t / n$
- Step 2. Calculate a – the "forward drift" and u – the "up-step"

$$a = e^{(r-q)\Delta t}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

Binomial pricing

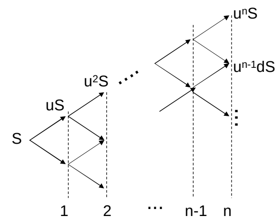
- Step 3. Calculate d – the “down-step” and p – the probability of an up move.

$$d = 1/u$$

$$p = \frac{a-d}{u-d}$$

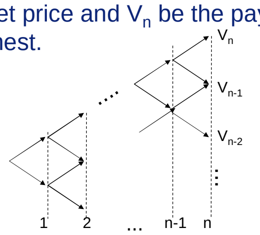
Binomial pricing

- Step 4. Evolve the asset price lattice.



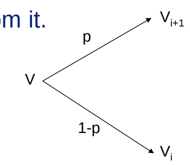
Binomial pricing

- Step 5. Calculate the payoff of the option V_i at each of the n possible expiry asset prices. Let V_1 be the payoff at the lowest asset price and V_n be the payoff at the highest.



Binomial pricing

- Step 6. Step back one time step and value each node based on the values of the two nodes that lead from it.



- V is given by the following formula

$$V = \left[pV_{i+1} + (1-p)V_i\right]e^{-r\Delta t}$$

Binomial pricing

- Step 7. Repeat the process until the leftmost node is reached. The value at that node is the option value.
- To value American options modify step 6

$$V = \max\left(\left[pV_{i+1} + (1-p)V_i\right]e^{-r\Delta t}, E\right)$$

- where E is the option payoff were it exercised at that node (i.e. we choose the higher of the exercise and "carrying" values).

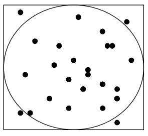
Monte-Carlo pricing

- Both analytic and binomial / finite difference methods are limited in the number of assets that can underly an option.
- Monte-Carlo can cope with many underlying assets (at least for European options).
- Monte-Carlo valuation of American options is more difficult.

Monte-Carlo pricing

- Monte-Carlo is a way of using random numbers to solve mathematical problems to do with area and integration.
- Take a non-financial example.
- Suppose we want to calculate the area of a circle but we don't know the formula for the area.

Monte-Carlo pricing



- Enclose the circle with a square.
- Generate random pairs of x-y co-ordinates.
- Count the number that are within the circle.
- Then the area of the circle is approximately the area of the square times the proportion of dots within the circle.

Monte-Carlo pricing

- Consider an option on several assets (e.g. "best-of", "exchange", "outperformance", etc)
- Step 1. Decompose the correlation matrix.
  - Suppose we have three assets and the correlation between their returns is described by the following matrix.

	Asset correlations		
	#1	#2	#3
#1	100%	30%	40%
#2	30%	100%	50%
#3	40%	50%	100%

  - Call the correlation matrix C

Monte-Carlo pricing

- Step 1 (cont)
  - We need to find a matrix A such that A multiplied by its transpose A^T equals C
  - i.e. $A.A^T = C$
  - There are various techniques for doing this.
  - We'll use "Cholesky decomposition"
  - Cholesky decomposition isn't an intrinsic spreadsheet function so it can be implemented as a Visual Basic function.
  - The following slide shows suitable VBA code

Monte-Carlo pricing

```
Function Cholesky(mat)
   Dim A As Variant, L() As Double, S As Double, n As Integer, m As Integer, j As Integer, k As Integer, i As Integer
   Dim A = mat
   If TypeName(mat) = "Range" Then
      n = mat.Rows.Count : m = mat.Columns.Count
   Else
      n = UBound(mat) : m = UBound(mat, 2)
   End If
   If n <> m Then
      Cholesky = "?“: Exit Function
   End If
   ReDim L(1 To n, 1 To n)
   For j = 1 To n
      S = 0
      For k = 1 To j - 1
         S = S + L(j, k) ^ 2
      Next k
      L(j, j) = A(j, j) - S
      If L(j, j) <= 0 Then Exit For
      L(j, j) = Sqr(L(j, j))
      For i = j + 1 To n
         S = 0
         For k = 1 To j - 1
            S = S + L(i, k) * L(j, k)
         Next k
         L(i, j) = (A(i, j) - S) / L(j, j)
      Next i
   Next j
   Cholesky = L
End Function
```


Monte-Carlo pricing

- Step 1 (cont)
  - The decomposed matrix is shown below

	A	B	C	D	E
1		Asset correlations			
2		#1	#2	#3	
3	#1	100%	30%	40%	
4	#2	30%	100%	50%	
5	#3	40%	50%	100%	
6					
7		Decomposed matrix -			
8		#1	#2	#3	
9	#1	100%	0%	=cholesky(B3:D5)	
10	#2	30%	95%	0%	
11	#3	40%	40%	83%	

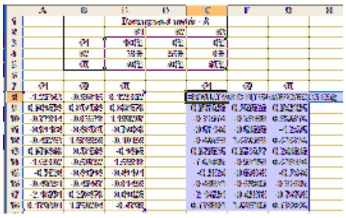
Monte-Carlo pricing

- Step 2. Check that the decomposed matrix times its transpose equals the original.
- Step 3. Generate three columns of normally distributed random numbers by using NORMSINV(RAND())

#1	#2	#3
-1.22343	-0.90815	0.129022
0.501536	0.634386	0.096579
-0.77914	-0.03678	1.198207
-0.91164	-0.66363	-0.74009
-0.42263	1.676999	-0.10199
0.575689	0.36886	-0.1655
-1.69002	-0.50792	1.56918
-0.3824	-0.61216	-0.41141
-0.48391	-0.42447	-0.01499
-2.16231	0.220679	0.01095
0.178301	1.250204	-0.4789

Monte-Carlo pricing

- Step 4. Multiply the columns of random numbers by the transpose of the decomposed correlation matrix



Monte-Carlo pricing

- Step 5. Evolve the assets time-step by time-step by using the following formula:

$$P_{k,i+1} = P_{k,i} \exp\left((r - q_k - \sigma_k^2 / 2)\Delta t + Z_{k,i}\sigma_k \sqrt{\Delta t}\right)$$

- $P_{k,i}$ is the evolved price for asset k at time interval i.
- σ_k is asset k's volatility
- q_k is asset k's yield
- $Z_{k,i}$ is the i^{th} random number for asset k as generated in step 4

Monte-Carlo pricing

- Now that the asset prices have been evolved the option payoff can be determined.
- Then the process is repeated from step 3 and individual option prices averaged.
- The average gives the Monte-Carlo estimate of the option's premium.